Math 2050, summary of Week 8

1. Ordering and convergence

We are always working on the following situation: $A \subset \mathbb{R}$ and c is a cluster point of A. We have known that the limit of function at cluster points have similar properties as the limit of sequence. We also have the following related to the ordering.

Theorem 1.1. *Let $f, g, h : A \to \mathbb{R}$, if*

$$f(x) \le g(x) \le h(x)$$

for all $x \in A$, then

1. *if $\lim_{x\to c} f = F$, $\lim_{x\to c} g = G$ and $\lim_{x\to c} h = H$, we have*
$$F \le G \le H.$$
2. *if $\lim_{x\to c} f = \lim_{x\to c} h = L$, then g has a limit as $x \to c$. In particular the limit is L.*

The importance of this result to that whenever we have well-behaved competitor, we can determine the convergence at some particular point.

Examples:

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ using $x - \frac{1}{6}x^3 \le \sin x \le x$ for all $x \ge 0$.
2. $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$ using $-\frac{1}{2}x^2 \le \cos x - 1 \le 0$ for $x > 0$.
3. $\lim_{x\to 0} x\sin(1/x) = 0$ using $|x \sin x| \le |x|$.

2. Some variation of limits

2.1. **One sided limits.** Consider the example

$$f(x) = \begin{cases} e^{1/x}, & \text{if } x \ne 0; \\ 0, & \text{if } x = 0; \end{cases} \tag{2.1}$$

Then the function has different behaviour when x tends to 0 from different directions. It is sometimes more important to consider one particular direction rather than all directions.

Definition 2.1. *Let $A \subset \mathbb{R}$ and c is a cluster point of $A \cap (c, +\infty)$, $f : A \to \mathbb{R}$. We say that $\lim_{x\to c^+} f = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that if $x \in A$ where $0 < x - c < \delta$, then*

$$|f(x) - L| < \varepsilon.$$

(similar for the left hand limit)

As expected from the theory for function, we have the following characterization using sequence which will be more user friendly when we discuss the divergence.

Theorem 2.1. *Let $A \subset \mathbb{R}$ and c is a cluster point of $A \cap (c, +\infty)$, $f : A \to \mathbb{R}$. Then $\lim_{x\to c^+} f = L$ if and only if for any $x_n \in A \cap (c, +\infty)$ where $x_n \to c$, we have $f(x_n) \to L$.*

Example: The function f defined at the beginning have $\lim_{x\to 0^-} f = 0$ but has no limit as $x \to 0^+$.

2.2. **Infinite limit.** The behaviour of f defined above as $x \to 0^+$ is divergent, but its divergence is relatively well-behaved as $f(x) \to +\infty$.

Definition 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. We say that $\lim_{x\to c} f = +\infty$ if $\forall \alpha > 0$, there is $\delta > 0$ such that if $x \in A$ such that $0 < |x - c| < \delta$, then $f(x) > \alpha$.*

One might compare this with the sequence.

Definition 2.3. *A sequence $\{a_n\}$ is said to be divergent to $+\infty$ if $\forall \alpha > 0, \exists N \in \mathbb{N}$ such that for all $n > N$, $a_n > \alpha$.*

The sequence criterion can be formulated similarly.

Theorem 2.2. *Let $A \subset \mathbb{R}$ and c is a cluster point of A, $f : A \to \mathbb{R}$. Then $\lim_{x\to c} f = +\infty$ if and only if for any $x_n \in A \setminus \{c\}$ where $x_n \to c$, we have $f(x_n) \to +\infty$.*

2.3. **limit at infinity.** The example mentioned above has certain decay properties as $x \to \infty$. To make it rigorous, we have

Definition 2.4. *Let $A \subset \mathbb{R}$ and suppose $(a, +\infty) \subset A$ for some $a \in \mathbb{R}$, and $f : A \to \mathbb{R}$. We say that $\lim_{x\to+\infty} f = L$ if $\forall \varepsilon > 0$, there $\alpha \in \mathbb{R}$ such that for all $x > \alpha$,*

$$|f(x) - L| < \varepsilon.$$

Similarly, we have

Definition 2.5. *Let $A \subset \mathbb{R}$ and suppose $(a, +\infty) \subset A$ for some $a \in \mathbb{R}$, and $f : A \to \mathbb{R}$. We say that $\lim_{x\to+\infty} f = +\infty$ if $\forall \beta > 0$, there $\alpha \in \mathbb{R}$ such that for all $x > \alpha$,*

$$f(x) > \beta.$$

Example:

(1) $\lim_{x\to+\infty} x^m = +\infty$ for all $m \in \mathbb{N}$;
(2) $\lim_{x\to+\infty} p(x) = +\infty$ if $p(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n > 0$;
(3) $\lim_{x\to+\infty} e^{-x} = 0$.

3. Continuous function

Recall that to consider the limit $\lim_{x\to c} f = L$, we allow the situation:

$$f(x) = \begin{cases} x, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0; \end{cases} \tag{3.1}$$

It is clear that $f(0) \neq \lim_{x\to 0} f$. But the limit is still well-behaved. (That is why we only consider $x \in A, 0 < |x-c| < \delta$ in the definition without considering $x = c$.)

We now pay more attention to the case when f is continuous. Want to rule out the above situation!

Definition 3.1. *Let $A \subset \mathbb{R}, c \in A$ and $f : A \to \mathbb{R}$. We say that f is continuous at c if $\forall \varepsilon > 0, \exists \delta > 0$ such that for all $x \in A$, $|x-c| < \delta$, we have*

$$|f(x) - f(c)| < \varepsilon.$$

Remark:

(1) If c is a cluster point, then the continuity implies i) $c \in A$; ii) f has limit at c; iii)$\lim_{x\to c} f = f(c)$.
(2) if c is not a cluster point, then there is $\delta > 0$ such that $A \cap \{y : 0 < |y-c| < \delta\} = \emptyset$. Hence, we always have the continuity f at c.

In term of sequence criterion:

Theorem 3.1. *Let $A \subset \mathbb{R}$ and $c \in A$, $f : A \to \mathbb{R}$. Then $\lim_{x\to c} f = f(c)$ if and only if for any $x_n \in A$ where $x_n \to c$, we have $f(x_n) \to f(c)$.*

Example:

(a)

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Then f is discontinuous at any point $c \in \mathbb{R}$. If $c \in \mathbb{Q}$, then $f(c) = 1$. But we can take $x_n \notin \mathbb{Q}$ by density of $\mathbb{Q}'$ such that $x_n \to c$ so that $f(x_n) \equiv 0$. Similarly, if $c \notin \mathbb{Q}$, we have similar contradicting sequence.

(b)

$$f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ where } gcd(m,n) = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Then f is discontinuous at $c \in \mathbb{Q}$ and is continuous at $c \notin \mathbb{Q}$. Discontinuous is similar to the first example. We may assume $c > 0$.

To show the continuity, for $\varepsilon > 0$, fix N such that $N^{-1} < \varepsilon$. Consider the set $\{(m, n) : n \leq N\}$, if $|mn^{-1} - c| < c$, we have

$$m \leq 2cn \leq 2cN.$$

Therefore, there is only finitely many element in form of $x = mn^{-1}$ so that $|x - c| < 1$ and $n \leq N$. Since $c \notin \mathbb{Q}$, c is not one of the element. Hence, c is isolated from the set $\{mn^{-1} : n \leq N\} \cap \{x : |x - c| < 1\}$. Therefore, we can find $\delta > 0$ such that

$$B = \{x : |x - c| < \delta\} \cap \{mn^{-1} : n \leq N\} = \emptyset.$$

Hence, for $x \in B$, we have $f(x) = \frac{1}{n} < \frac{1}{N} < \varepsilon$.